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Spontaneous Lorentz Symmetry Breaking by Anti-Symmetric Tensor Field

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Abstract

We study the spontaneous Lorentz symmetry breaking in a field theoretical model in (2+1)-dimension, inspired by string theory. This model is a gauge theory of an anti-symmetric tensor field and a vector field (photon). The Nambu-Goldstone (NG) boson for the spontaneous Lorentz symmetry breaking is identified with the unphysical massless photon in the covariant quantization. We also discuss an analogue of the equivalence theorem between the amplitudes for emission or absorption of the physical massive anti-symmetric tensor field and those of the unphysical massless photon. The low-energy effective action of the NG-boson is also discussed.

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1 Introduction

Quantum field theories based on the Poincaré invariance, in particular, the Lorentz invariance successfully describe elementary particles below the weak scale energy (~ 100 GeV). In a last few years, a possible type of the Lorentz *non*-invariant extensions of the quantum field theories has been extensively studied. These are the field theories on the space-time whose coordinates are non-commutative, called the non-commutative field theories[1, 2, 3, 4]. The action of the non-commutative field theories can be constructed by replacing the product of fields in the action of the ordinary field theory with the \star -product defined as

$$f(x) \star g(x) \equiv e^{i\theta^{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \eta^j}} f(x + \xi) g(x + \eta) \Big|_{\xi=\eta=0}, \quad (1.1)$$

where θ^{ij} is a constant non-commutative parameter: $[x^i, x^j] = i\theta^{ij}$. Thus the action explicitly contains the *constant* anti-symmetric tensor θ^{ij} , and the Lorentz invariance in (p+1)-dimension for $p \geq 2$ cannot be maintained.

String theory naturally provides the non-commutative field theories as the world volume effective theories on D-branes[4]: the world volume effective theory of Dp-brane with a constant background NS-NS B-field is equivalent to a (p+1)-dimensional non-commutative field theory whose constant non-commutative parameter θ^{ij} is given by the background NS-NS B-field B_{ij} . In string theory the NS-NS B-field is indeed a dynamical field in closed string sector and thus the constant background field can be interpreted as the constant vacuum expectation value of the dynamical NS-NS B-field. From this perspective, the Lorentz symmetry is *spontaneously* broken by the constant vacuum expectation value of the second rank anti-symmetric tensor field.

In this paper, based on this viewpoint, we discuss the spontaneous Lorentz symmetry breaking within the effective field theory of the string theory. Concretely, we investigate the Nambu-Goldstone boson for the Lorentz symmetry breaking in a field theoretical toy model in (2+1)-dimension of a second rank anti-symmetric tensor field and a vector field, which is inspired by the effective theory of the string theory. We find that the NG-boson is an unphysical field and their amplitudes, however, provide the useful information about the physical amplitudes of the model through the “equivalence theorem”. We also discuss the low-energy dynamics of the NG-boson from the perspective of the nonlinear realization of the Lorentz symmetry.

This paper is organized as follows. In the next section, we introduce the gauge invariant model of a second rank anti-symmetric tensor field and a vector field and discuss the covariant canonical quantization of the model. In section 3, the vacuum of the model

where the anti-symmetric tensor field has a constant vacuum expectation value is discussed and also the Nambu-Goldstone boson for the spontaneous Lorentz symmetry breaking is studied in detail. In section 4, a possible perturbation of the model is discussed and the equivalence theorem between the physical amplitudes and the amplitudes of the unphysical NG-boson is also argued. In section 5, some related problems are discussed and the relation to the non-commutative field theories is speculated.

2 A toy model for field theory of $B_{\mu\nu}$ and A_μ

In this section we discuss the covariant canonical quantization of a toy model for the gauge invariant field theory of an second rank anti-symmetric tensor field $B_{\mu\nu}$ coupled with a vector field A_μ (photon) in (2+1)-dimension.

2.1 Canonical quantization of the model

The action of the toy model is given by³

$$S = \int d^3x \left(\frac{1}{12m^2} (H_{\mu\nu\rho})^2 - \frac{1}{4} (F_{\mu\nu} - B_{\mu\nu})^2 \right), \quad (2.1)$$

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.2)$$

and m is a parameter with dimension of mass. This action is inspired by string theory⁴. Indeed, the first and second term in (2.1) are the same form as the leading term of the effective action of $B_{\mu\nu}$, which is a massless mode of closed string, and the leading term of the Dirac-Born-Infeld (DBI) action of D-brane world volume effective theory, which is the effective action of the open string sector, in α' -expansion[9].

The action (2.1) is invariant under the gauge transformation:

$$\begin{aligned} \delta B_{\mu\nu}(x) &= \partial_\mu \Lambda_\nu(x) - \partial_\nu \Lambda_\mu(x), \\ \delta A_\mu(x) &= \Lambda_\mu(x) + \partial_\mu \Lambda(x), \end{aligned} \quad (2.3)$$

where $\Lambda_\mu(x)$ and $\Lambda(x)$ are 1-form and scalar gauge functions respectively. Because of this gauge invariance, the system described by the action (2.1) is a singular (constrained) system. Thus, for the canonical quantization, one must introduce gauge fixing terms.

³The metric is $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag.}(+1, -1, -1)$.

⁴In fact, this type of action appears in various contexts of string theory[5, 6, 7, 8].

Since we want to discuss the spontaneous Lorentz symmetry breaking in the sequel, we must take a Lorentz invariant gauge fixing terms. We introduce the following gauge fixing terms:

$$S_{\text{gf}} = \int d^3x \left(C^\nu \partial^\mu B_{\mu\nu} - B \partial_\mu A^\mu - C \partial_\mu C^\mu \right), \quad (2.4)$$

where $B(x)$ is the Nakanishi-Lautrup (NL) B-field for the vector field and $C_\mu(x)$ and $C(x)$ are the counterparts for the anti-symmetric tensor gauge field[10, 11]. These gauge fixing terms are the analogues of the Landau gauge in quantum electrodynamics (QED).

Although the canonical quantization of the model in the BRST formalism can be carried out, we make the canonical quantization in the NL formalism[10, 11] for simplicity⁵.

The gauge fixed action is given by (2.1) and (2.4):

$$S_{\text{total}} = \int d^3x \mathcal{L}_{\text{total}}(x) = \int d^3x \left(\frac{1}{12m^2} (H_{\mu\nu\rho})^2 - \frac{1}{4} (F_{\mu\nu} - B_{\mu\nu})^2 + C^\nu \partial^\mu B_{\mu\nu} - B \partial_\mu A^\mu - C \partial_\mu C^\mu \right). \quad (2.5)$$

The equations of motion derived from (2.5) for each field become as follows.

$$B_{\mu\nu} : \frac{1}{m^2} \partial_\rho H^{\rho\mu\nu} + (B^{\mu\nu} - F^{\mu\nu} + C^{\mu\nu}) = 0, \quad (2.6)$$

$$A_\mu : -\partial_\rho (F^{\rho\mu} - B^{\rho\mu}) - \partial^\mu B = 0, \quad (2.7)$$

$$C_\mu : -\partial_\rho B^{\rho\mu} - \partial^\mu C = 0, \quad (2.8)$$

$$B : \partial_\mu A^\mu = 0, \quad (2.9)$$

$$C : \partial_\mu C^\mu = 0, \quad (2.10)$$

where $C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. Actually, by combining these equations, one can find free field equations of each field:

$$\begin{aligned} \square^2 A_\mu &= 0, & \square (\square + m^2) B_{\mu\nu} &= 0, \\ \square^2 C_\mu &= 0, & \square B &= 0, & \square C &= 0. \end{aligned} \quad (2.11)$$

Thus this model is essentially a free field theory and can be quantized completely. Note that the anti-symmetric tensor field $B_{\mu\nu}$ is a mixture of massive and massless components.

⁵In the BRST formalism, ghost and anti-ghost fields are introduced in addition. However, since this action is a quadratic action with abelian gauge symmetry, ghost and anti-ghost fields are free fields and decouple

Following the procedure in [10, 11], the three-dimensional commutation relations can be calculated by using the equal-time commutation relations,

$$\begin{aligned} [\phi_I(\mathbf{x}, t), \phi_J(\mathbf{y}, t)] &= 0, & [\pi^I(\mathbf{x}, t), \pi^J(\mathbf{y}, t)] &= 0, \\ [\phi_I(\mathbf{x}, t), \pi^J(\mathbf{y}, t)] &= i\delta_I^J \delta^2(\mathbf{x} - \mathbf{y}), \quad \text{where } \pi^I(\mathbf{x}, t) \equiv \frac{\partial \mathcal{L}_{\text{total}}(x)}{\partial \dot{\phi}_I(\mathbf{x}, t)}. \end{aligned} \quad (2.12)$$

Here we abbreviate various field as $\phi_I(\mathbf{x}, t)$, where I denotes various indices. The explicit forms of the non-vanishing three-dimensional commutation relations are⁶

$$\begin{aligned} [B(x), A_\mu(y)] &= [C(x), C_\mu(y)] = -i\partial_\mu^x D(x - y), \\ [A_\mu(x), A_\nu(y)] &= [C_\mu(x), A_\nu(y)] = -i\eta_{\mu\nu} D(x - y) + i\partial_\mu^x \partial_\nu^x E(x - y), \\ [C_\mu(x), B_{\nu\rho}(y)] &= -i(\eta_{\mu\nu} \partial_\rho^x - \eta_{\mu\rho} \partial_\nu^x) D(x - y), \\ [B_{\mu\nu}(x), B_{\rho\sigma}(y)] &= i\left(\eta_{\mu\rho} \partial_\nu^x \partial_\sigma^x - \eta_{\mu\sigma} \partial_\nu^x \partial_\rho^x - \eta_{\nu\rho} \partial_\mu^x \partial_\sigma^x + \eta_{\nu\sigma} \partial_\mu^x \partial_\rho^x \right. \\ &\quad \left. + m^2(\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho})\right) \Delta(x - y : m^2) \\ &\quad - i\left(\eta_{\mu\rho} \partial_\nu^x \partial_\sigma^x - \eta_{\mu\sigma} \partial_\nu^x \partial_\rho^x - \eta_{\nu\rho} \partial_\mu^x \partial_\sigma^x + \eta_{\nu\sigma} \partial_\mu^x \partial_\rho^x\right) D(x - y), \end{aligned} \quad (2.13)$$

where

$$\Delta(x : m^2) \equiv \frac{1}{(2\pi)^2 i} \int d^3k \, \epsilon(k_0) \delta(k^2 - m^2) e^{-ikx}, \quad D(x) \equiv \Delta(x : m^2 = 0), \quad (2.14)$$

$$E(x) \equiv \frac{1}{(2\pi)^2 i} \int d^3k \, \epsilon(k_0) \delta'(k^2) e^{-ikx}, \quad \square E(x) = D(x). \quad (2.15)$$

In order to quantize the model consistently, we require the physical state conditions analogous to the ordinary QED in the NL formalism[10, 11]. We define the *physical state* through the physical state conditions:

$$C_\mu^{(+)}(x)|\text{phys}\rangle = 0, \quad B^{(+)}(x)|\text{phys}\rangle = 0, \quad C^{(+)}(x)|\text{phys}\rangle = 0, \quad (2.16)$$

where $\phi_I^{(+)}(x)$ means the positive energy part of $\phi_I(x)$. In the gauge (2.4), as seen from (2.11), $C_\mu(x)$ is a dipole field. Although the separation between the positive and negative energy part of a dipole field is a non-trivial problem, the cut-off procedure is known to give the well-defined separation as is found in the next subsection[11, 12]. Thus the physical state conditions (2.16) are well-defined.

⁶The equal-time commutation relations are obtained by setting $x^0 = y^0$ in the three-dimensional commutation relations.

2.2 The physical spectrum

In order to find the spectrum of the model, we define the creation and annihilation operators of each field. The annihilation operators are defined by the Fourier transforms:

$$\begin{aligned}
C^{(+)}(x) &= \frac{1}{2\pi} \int d^3k \theta(k_0) e^{-ikx} b(k), \\
B^{(+)}(x) &= \frac{1}{2\pi} \int d^3k \theta(k_0) e^{-ikx} c(k), \\
C_\mu^{(+)}(x : \epsilon) &= \frac{1}{2\pi} \int d^3k \theta(k_0 - \epsilon) e^{-ikx} c_\mu(k), \\
A_\mu^{(+)}(x : \epsilon) &= \frac{1}{2\pi} \int d^3k \theta(k_0 - \epsilon) e^{-ikx} a_\mu(k), \\
B_{\mu\nu}^{(+)}(x) &= \frac{1}{2\pi} \int d^3k \theta(k_0) e^{-ikx} b_{\mu\nu}(k),
\end{aligned} \tag{2.17}$$

and the creation operators are defined by the hermitian conjugate of (2.17). ϵ in the definitions (2.17) is an infra-red cut-off parameter for the dipole fields[11, 12].

The commutation relations of the operators can be calculated by the three-dimensional commutation relations (2.13). The non-vanishing commutation relations are

$$\begin{aligned}
[b(p), a_\mu^+(k)] &= [c(p), c_\mu^+(k)] = ip_\mu \theta(p_0) \delta(p^2) \delta^3(p - k), \\
[a_\mu(p), a_\nu^+(k)] &= [c_\mu(p), a_\nu^+(k)] = -\eta_{\mu\nu} \theta(p_0) \delta(p^2) \delta^3(p - k) - p_\mu p_\nu \theta(p_0) \delta'(p^2) \delta^3(p - k), \\
[c_\mu(p), b_{\nu\rho}^+(k)] &= i(\eta_{\mu\nu} p_\rho - \eta_{\mu\rho} p_\nu) \theta(p_0) \delta(p^2) \delta^3(p - k), \\
[b_{\mu\nu}(p), b_{\rho\sigma}^+(k)] &= (-\eta_{\mu\rho} p_\nu p_\sigma + \eta_{\mu\sigma} p_\nu p_\rho + \eta_{\nu\rho} p_\mu p_\sigma - \eta_{\nu\sigma} p_\mu p_\rho \\
&\quad + m^2 (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho})) \theta(p_0) \delta(p^2 - m^2) \delta^3(p - k) \\
&\quad - (-\eta_{\mu\rho} p_\nu p_\sigma + \eta_{\mu\sigma} p_\nu p_\rho + \eta_{\nu\rho} p_\mu p_\sigma - \eta_{\nu\sigma} p_\mu p_\rho) \theta(p_0) \delta(p^2) \delta^3(p - k).
\end{aligned} \tag{2.18}$$

In terms of these operators, the physical state conditions (2.16) become

$$c_\mu(p)|\text{phys}\rangle = 0, \quad b(p)|\text{phys}\rangle = 0, \quad c(p)|\text{phys}\rangle = 0. \tag{2.19}$$

The vacuum state is defined by

$$b_{\mu\nu}(p)|\text{vac}\rangle = 0, \quad a_\mu(p)|\text{vac}\rangle = 0, \tag{2.20}$$

$$c_\mu(p)|\text{vac}\rangle = 0, \quad b(p)|\text{vac}\rangle = 0, \quad c(p)|\text{vac}\rangle = 0. \tag{2.21}$$

The vacuum state is physical by definition. One particle states are constructed by the creation operators from the vacuum state. Physical one particle states, which satisfy the conditions (2.19), are constructed by the creation operators which commute with $c_\mu(p)$, $b(p)$, and $c(p)$. The physical states are summarized as follows⁷.

⁷ $f_{\mu\nu}(p)$ and $c_{\mu\nu}(p)$ are the Fourier transforms of $F_{\mu\nu}^{(+)}(x : \epsilon)$ and $C_{\mu\nu}^{(+)}(x : \epsilon)$, respectively.

(i) Physical massless states in the momentum frame $p_\mu = (p, 0, p)$,

$$c_1^+(p)|\text{vac}\rangle, \quad (c_0^+(p) - c_2^+(p))|\text{vac}\rangle, \quad b^+(p)|\text{vac}\rangle. \quad (2.22)$$

(ii) Physical massive states with mass m in the rest frame $p_\mu = (m, 0, 0)$,

$$u_{\mu\nu}^+(p)|\text{vac}\rangle \equiv (b_{\mu\nu}^+(p) - f_{\mu\nu}^+(p) + c_{\mu\nu}^+(p))|\text{vac}\rangle. \quad (2.23)$$

Here $u_{\mu\nu}^+(p)$ is the creation operator of the gauge invariant field $U_{\mu\nu}(x) \equiv B_{\mu\nu}(x) - F_{\mu\nu}(x) + C_{\mu\nu}(x)$. Note that the massless states of photon $a_\mu(p)$ are all unphysical due to the “large” gauge symmetry with 1-form gauge function (2.3).

One can show that all the physical massless states (2.22) are null states from the commutation relations (2.18). The physical massive state $u_{12}^+(p)|\text{vac}\rangle$ is the propagating states with positive norm and $u_{01}^+(p)|\text{vac}\rangle$ and $u_{02}^+(p)|\text{vac}\rangle$ are null states in the rest frame. Thus we conclude that the physical propagating degree of freedom⁸ of the model is a physical massive state $u_{12}^+(p)|\text{vac}\rangle$ with mass m . Although the action (2.1) has the gauge symmetry (2.3), the physical massive state appears through the generalized Stueckelberg formalism, which is the anti-symmetric tensor field version of the Stueckelberg formalism of QED[13]. The anti-symmetric tensor field $B_{\mu\nu}$ “eats” the degrees of freedom of the gauge field A_μ and become a massive anti-symmetric tensor field. Note that massless second rank anti-symmetric tensor field has no physical propagating degrees of freedom and massive one has one physical propagating degree in (2+1)-dimension⁹.

We consider the interesting limit of the model, $m \rightarrow 0$. This corresponds to the limit where the modes of closed string decouple in the corresponding effective action of string theory discussed in the previous subsection. In this limit, the commutation relation of $B_{\mu\nu}$ in (2.13) becomes

$$[B_{\mu\nu}(x), B_{\rho\sigma}(y)] = 0, \quad (2.24)$$

and the commutators of the creation and annihilation operators also become

$$[b_{\mu\nu}(p), b_{\rho\sigma}^+(k)] = 0. \quad (2.25)$$

Thus the states associated with the anti-symmetric tensor field $B_{\mu\nu}$ become zero norm. In this limit, the physical propagating massless state in the momentum frame $p_\mu = (p, 0, p)$ is given by

$$(u_{01}^+(p) - u_{12}^+(p))|\text{vac}\rangle. \quad (2.26)$$

⁸In this paper, the physical propagating state means the physical state with positive norm which contributes to the physical amplitudes.

⁹Massless photon has also one physical propagating degree in (2+1)-dimension.

Indeed, the norm of this physical propagating state becomes

$$\begin{aligned} \left\langle (u_{01}(p) - u_{12}(p)) (u_{01}^+(p) - u_{12}^+(p)) \right\rangle &= \left\langle (f_{01}(p) - f_{12}(p)) (f_{01}^+(p) - f_{12}^+(p)) \right\rangle \\ &= 4p^2 \left\langle a_1(p) a_1^+(p) \right\rangle. \end{aligned} \quad (2.27)$$

Hence, in this limit, the physical propagating state becomes essentially the transverse photon $a_1^+(p)|\text{vac}\rangle$. However it is worth noting that even though the norms of $(u_{01}^+(p) - u_{12}^+(p))|\text{vac}\rangle$ and $(f_{01}^+(p) - f_{12}^+(p))|\text{vac}\rangle$ are same in the limit $m \rightarrow 0$, the physical propagating state is *not* $(f_{01}^+(p) - f_{12}^+(p))|\text{vac}\rangle$, *but* $(u_{01}^+(p) - u_{12}^+(p))|\text{vac}\rangle$.

This situation is similar to the broken phase of Yang-Mills-Higgs model, where the equivalence theorem holds[14, 15]. This theorem claims the amplitude for emission or absorption of the longitudinal states of the massive gauge boson becomes equal, at high energy, to the amplitude for emission or absorption of the unphysical Nambu-Goldstone states, which is “eaten” by the gauge boson. In our model, the physical massive state of the anti-symmetric tensor field appears after the anti-symmetric tensor field “eats” the unphysical state of the transverse photon. The above analysis of the norm of the physical states implies that the analogous equivalence theorem holds in our model: in the high energy region where one can ignore mass m , the amplitude for emission or absorption of the longitudinal states of the physical massive anti-symmetric tensor field is the same as the amplitude for emission or absorption of the unphysical massless transverse photon.

3 Spontaneous Lorentz symmetry breaking by anti-symmetric tensor field

In this section we discuss the spontaneous breaking of the Lorentz symmetry by a constant vacuum expectation value (vev) of the second rank anti-symmetric tensor field in our model.

3.1 The Nambu-Goldstone boson for the spontaneous Lorentz symmetry breaking

The equations of motion (2.6)-(2.10) have a solution such that only $B_{\mu\nu}$ and $F_{\mu\nu}$ have constant nonzero vev's¹⁰:

$$\langle B_{12} \rangle = \langle F_{12} \rangle = B_{\text{vev}} = \text{const.} (\neq 0), \quad (3.1)$$

¹⁰If the vev $\langle \tilde{B}^\mu \rangle$ ($\equiv \frac{1}{2}\epsilon^{\mu\nu\rho}\langle B_{\nu\rho} \rangle$) = $\langle \tilde{F}^\mu \rangle$ is a time-like constant vector, one can transform it to this form by an appropriate Lorentz transformation.

One can easily find that this solution is a ground state with vanishing energy of the Hamiltonian derived from the action (2.5). (See (3.3).) Although B_{vev} is an undetermined constant in our model, we assume this to be a nonzero constant in the sequel. Since the vev's of C_μ , B , and C vanish, this vacuum is a physical state. In frameworks of the nonperturbative string theory, the possibility of $\langle B_{\mu\nu} \rangle \neq 0$ has been discussed in [16, 17]. In this viewpoint, the nonzero vev of B_{12} induces a spontaneous magnetization $\langle F_{12} \rangle$.

Since $B_{\mu\nu}$ and $F_{\mu\nu}$ are not gauge invariant under the gauge transformation (2.3), one may expect to be able to eliminate this vev by the gauge transformation. However the gauge transformation which eliminates the vev requires the linear 1-form gauge functions such as $\Lambda_1(x) = \frac{1}{2}B_{\text{vev}}x^2$ and $\Lambda_2(x) = -\frac{1}{2}B_{\text{vev}}x^1$. These gauge functions are ill-defined at the infinity. Hence we do not require the invariance under such singular gauge transformations to define the Hilbert space of the quantum theory.

As discussed in [18], on the vacuum (3.1), the (2+1)-dimensional Lorentz symmetry $SO(2, 1) \sim SL(2, \mathbf{R})$ is spontaneously broken down to the spatial rotation $SO(2) \sim U(1)$ by the vev of B_{12} and F_{12} ¹¹. What is the Nambu-Goldstone (NG) bosons for the broken boost generators in this model ?

In order to answer this question, we construct the generators of the Lorentz transformations $M_{\rho\sigma}$ by the Noether method. The conserved currents $M_{\rho\sigma}{}^\mu(x)$ for the Lorentz symmetry can be derived from the action (2.5)¹²:

$$\begin{aligned} M_{\rho\sigma}{}^\mu(x) &= x_\rho T_\sigma{}^\mu(x) - x_\sigma T_\rho{}^\mu(x) - i \frac{\partial \mathcal{L}_{\text{total}}(x)}{\partial (\partial_\mu \phi_I)} (S_{\rho\sigma} \phi)_I \\ &= \frac{1}{2m^2} H^{\mu\alpha\beta} (x_\rho \partial_\sigma - x_\sigma \partial_\rho) B_{\alpha\beta} - (F^{\mu\alpha} - B^{\mu\alpha}) (x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\alpha \\ &\quad + C_\alpha (x_\rho \partial_\sigma - x_\sigma \partial_\rho) B^{\mu\alpha} - B (x_\rho \partial_\sigma - x_\sigma \partial_\rho) A^\mu - C (x_\rho \partial_\sigma - x_\sigma \partial_\rho) C^\mu \\ &\quad - (x_\rho \delta_\sigma^\mu - x_\sigma \delta_\rho^\mu) \left(\frac{1}{12m^2} (H_{\alpha\beta\gamma})^2 - \frac{1}{4} (F_{\alpha\beta} - B_{\alpha\beta})^2 + C^\beta \partial^\alpha B_{\alpha\beta} \right) \\ &\quad + \left(\frac{1}{m^2} H^{\mu\alpha\beta} + \eta^{\mu\alpha} C^\beta - \eta^{\mu\beta} C^\alpha \right) (\eta_{\rho\alpha} B_{\sigma\beta} - \eta_{\sigma\alpha} B_{\rho\beta}) \\ &\quad - (F^{\mu\alpha} - B^{\mu\alpha} + \eta^{\mu\alpha} B) (\eta_{\rho\alpha} A_\sigma - \eta_{\sigma\alpha} A_\rho) - \eta^{\mu\alpha} C (\eta_{\rho\alpha} C_\sigma - \eta_{\sigma\alpha} C_\rho), \quad (3.2) \end{aligned}$$

¹¹Our convention for the Poincaré algebra in (2+1)-dimension is

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \quad [M_{\mu\nu}, P_\rho] = -i (\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i (\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho}). \end{aligned}$$

¹² $(S_{\rho\sigma} \phi)_I$ denotes infinitesimal transformations of internal spin: $(S_{\rho\sigma} \chi) = 0$ for scalar fields, $(S_{\rho\sigma} V)_\mu = i (\eta_{\rho\mu} V_\sigma - \eta_{\sigma\mu} V_\rho)$ for vector fields, and $(S_{\rho\sigma} B)_{\mu\nu} = i (\eta_{\rho\mu} B_{\sigma\nu} - \eta_{\sigma\mu} B_{\rho\nu} + \eta_{\rho\nu} B_{\mu\sigma} - \eta_{\sigma\nu} B_{\mu\rho})$ for second rank anti-symmetric tensor fields.

where the canonical energy-momentum tensor $T_\rho^\mu(x)$ is given by

$$\begin{aligned} T_\rho^\mu(x) = & \frac{1}{2m^2} H^{\mu\alpha\beta} \partial_\rho B_{\alpha\beta} - (F^{\mu\alpha} - B^{\mu\alpha}) \partial_\rho A_\alpha \\ & + C_\alpha \partial_\rho B^{\mu\alpha} - B \partial_\rho A^\mu - C \partial_\rho C^\mu \\ & - \delta_\rho^\mu \left(\frac{1}{12m^2} (H_{\alpha\beta\gamma})^2 - \frac{1}{4} (F_{\alpha\beta} - B_{\alpha\beta})^2 + C^\beta \partial^\alpha B_{\alpha\beta} \right). \end{aligned} \quad (3.3)$$

From the conserved currents (3.2), one can obtain the generators of Lorentz transformation $M_{\rho\sigma}$:

$$M_{\rho\sigma} = \int d^2\mathbf{x} M_{\rho\sigma}{}^0(x). \quad (3.4)$$

By utilizing the expressions (3.2) and (3.4) and the commutation relations (2.13), we have the nonvanishing vev of the following commutation relations on the vacuum (3.1):

$$\langle [iM_{0i}, B_{0j}(x)] \rangle = \int d^2\mathbf{y} \langle [iM_{0i}{}^0(y), B_{0j}(x)] \rangle = \epsilon_{ij} B_{\text{vev}}, \quad (3.5)$$

$$\langle [iM_{0i}, F_{0j}(x)] \rangle = \int d^2\mathbf{y} \langle [iM_{0i}{}^0(y), F_{0j}(x)] \rangle = \epsilon_{ij} B_{\text{vev}} \quad (\epsilon_{12} = 1). \quad (3.6)$$

Thus two boost generators M_{0i} ($i = 1, 2$) are spontaneously broken on the vacuum. From the above commutation relations, the candidates for the NG-bosons for the broken boost generators are B_{0i} and F_{0i} . However, $B_{\mu\nu}$ is a mixture of massive and massless components as discussed previously. As obtained in the previous section, the mass eigenstates of the model are the massive field $U_{\mu\nu}$ and the massless field $F_{\mu\nu}$ (or A_μ) and the other massless fields C_μ , B , and C . Since NG-bosons are necessarily massless state, we conclude that the NG-bosons for the broken boost generators are the massless photon F_{0i} . Incidentally, the massive field $U_{\mu\nu}$ satisfies

$$\langle [iM_{0i}, U_{\mu\nu}(x)] \rangle = \int d^2\mathbf{y} \langle [iM_{0i}{}^0(y), U_{\mu\nu}(x)] \rangle = 0. \quad (3.7)$$

Thus the fact that $B_{\mu\nu}$ becomes essentially a massive field is consistent with the Nambu-Goldstone theorem.

3.2 Are the NG-bosons physical ?

We have identified the NG-bosons for the spontaneous Lorentz symmetry breaking with the massless photon F_{0i} . The one particle state of the massless photon $F_{\mu\nu}$ (or A_μ) is an unphysical state as discussed in the previous section. Hence the NG-bosons for

the spontaneous Lorentz symmetry breaking are unphysical states, i.e., unphysical NG-bosons. Since the Lorentz symmetry is a physical global symmetry, the corresponding NG-bosons are expected to be physical. Is this a contradiction ?

In order to answer this question, let us discuss the one particle state created by the broken boost generators from the vacuum (3.1). Inserting the decompositions $\phi_I(x) = \langle \phi_I \rangle + \hat{\phi}_I(x)$ on the vacuum¹³ into the expressions of the currents (3.2), one can construct the broken parts of the currents $M_{0i}^{B\mu}(x)$ which depends on B_{vev} , i.e., $M_{0i}^{B\mu} = M_{0i}^{B\mu}(B_{\text{vev}}, \hat{\phi}_I) + \hat{M}_{0i}^{B\mu}(\hat{\phi}_I)$. The explicit forms of the broken parts of the currents $M_{0i}^{B\mu}(x)$ are

$$M_{0i}^{B\mu}(x) = B_{\text{vev}} \epsilon_{ij} \left\{ \frac{1}{2} x^j \left(\hat{F}^{\mu 0} - \hat{B}^{\mu 0} + \eta^{\mu 0} \hat{B} \right) - \frac{1}{2} x_0 \left(\hat{F}^{\mu j} - \hat{B}^{\mu j} + \eta^{\mu j} \hat{B} \right) + \left(\frac{1}{m^2} \hat{H}^{\mu 0 j} + \eta^{\mu 0} \hat{C}^j - \eta^{\mu j} \hat{C}^0 \right) \right\} \quad (3.8)$$

and $M_{12}^{B\mu}$ vanishes. Using the equations of motion for $\hat{\phi}_I$ which are the same as (2.6)-(2.10), one can easily show that these currents are also conserved. From these currents one can obtain the broken parts of the generators which are conserved:

$$M_{0i}^B \equiv \int d^2 \mathbf{x} M_{0i}^{B0}(x) = B_{\text{vev}} \int d^2 \mathbf{x} \epsilon_{ij} \left\{ \left(\frac{1}{2} x_0 \hat{U}^{0j}(x) \right) + \left(-\frac{1}{2} x_0 \hat{C}^{0j}(x) + \frac{1}{2} x^j \hat{B}(x) + \hat{C}^j(x) \right) \right\}. \quad (3.9)$$

Hereafter we abbreviate simply $\hat{\phi}_I$ as ϕ_I without confusion. These broken parts of generators indeed satisfy

$$[iM_{0i}^B, B_{0j}(x)] = [iM_{0i}^B, F_{0j}(x)] = \epsilon_{ij} B_{\text{vev}}. \quad (3.10)$$

Note that M_{0i}^B is a physical operator which commutes with C_μ , B , and C .

We consider the one particle state created by M_{0i}^B from the vacuum (3.1), denoted as $|\text{VAC}\rangle$. Since we are interested in the NG-bosons for the broken generators, we consider only the massless state of the one particle state given by

$$M_{0i}^B |\text{VAC}\rangle \Big|_{\text{massless}} = B_{\text{vev}} \int d^2 \mathbf{x} \epsilon_{ij} \left(-\frac{1}{2} x_0 C^{0j}(x) + \frac{1}{2} x^j B(x) + C^j(x) \right) |\text{VAC}\rangle. \quad (3.11)$$

From the commutation relations (2.13), one can easily find that these states are the physical states which satisfy the conditions (2.16), but the null states. Therefore we conclude

¹³ Since $\langle F_{12} \rangle = B_{\text{vev}}$, we take, for an example, the vev's of A_i as $\langle A_1 \rangle = -\frac{1}{2} B_{\text{vev}} x^2$ and $\langle A_2 \rangle = \frac{1}{2} B_{\text{vev}} x^1$.

that although the spontaneous Lorentz symmetry breaking is *physical*, the massless one particle state created by the broken generator M_{0i}^B from the vacuum (3.1) is not only a physical state but also *null* state. Since our model has the twisted structure of the Hilbert space with an indefinite metric on account of the gauge invariance (2.3), nonzero matrix elements exist between physical null states and unphysical states. Thereby the NG-boson which has the nonzero matrix element with this massless one particle state can be *unphysical*. This is closely analogous to the abelian Higgs model in the NL formalism [11].

As discussed in the previous section, in the limit $m \rightarrow 0$, the unphysical transverse photon, which is the unphysical NG-boson for the spontaneous Lorentz symmetry breaking, becomes essentially the physical propagating state of the model. The implication of this fact will be discussed in the next section.

4 A perturbation

So far, we have studied the free field theory of $B_{\mu\nu}$ and A_μ whose action is given by (2.5). In this section, we discuss a possible perturbation of the model. We introduce the following interaction terms to (2.5) as the perturbation:

$$\mathcal{L}_{\text{int}}(x) = \sum_{n=2}^N a_n \{ (F_{\mu\nu} - B_{\mu\nu})^2 \}^n. \quad (4.1)$$

This type of interactions is obtained by the α' -expansion of DBI action¹⁴. These interactions are gauge invariant under the gauge transformation (2.3) and consistent with the physical state conditions (2.16): these interaction terms do not change the equations of motion of C_μ , B , and C in (2.11) and keep them free fields. Although these interactions are non-renormalizable, we treat them as the perturbations which are interpreted as the operator insertions in the matrix elements in the similar manner to the chiral Lagrangian of QCD.

Although the equations of motion of $B_{\mu\nu}$ and A_μ is modified due to the interaction terms, the solution (3.1) still remains to be a solution of the modified equations. Thus, even after including the perturbations, the spontaneous Lorentz symmetry breaking is realized. In this case, the argument about the NG-boson in the previous section does not change essentially and hence we can conclude that the NG-boson remains to be the unphysical photon.

¹⁴By virtue of the peculiarity of (2+1)-dimension, a Lorentz scalar constructed from $F_{\mu\nu} - B_{\mu\nu}$ can be always expressed as a polynomial of $(F_{\mu\nu} - B_{\mu\nu})^2$.

As is well-known, the low-energy dynamics of the NG-boson is given by only the symmetry argument, i.e., the low-energy theorem. In particular, the low-energy effective action of the NG-boson is given by the nonlinear realization of the broken symmetry. In the case of the spontaneous Lorentz symmetry breaking, these argument holds and the low-energy effective action of the NG-boson is expected to be given by the nonlinear realization of the broken Lorentz symmetry. This problem will be argued in the next section.

In the limit $m \rightarrow 0$, the physical propagating state becomes essentially the unphysical transverse photon. Furthermore the amplitude of the physical propagating state (2.26), for example, the two-point amplitude satisfies

$$\begin{aligned} & \left\langle (u_{01}(p) - u_{12}(p)) \tilde{\mathcal{L}}_{\text{int}}(q) (u_{01}^+(k) - u_{12}^+(k)) \right\rangle \\ &= \left\langle (f_{01}(p) - f_{12}(p)) \tilde{\mathcal{L}}_{\text{int}}^{(b_{\mu\nu}=0)}(q) (f_{01}^+(k) - f_{12}^+(k)) \right\rangle, \end{aligned} \quad (4.2)$$

where $\tilde{\mathcal{L}}_{\text{int}}(q)$ and $\tilde{\mathcal{L}}_{\text{int}}^{(b_{\mu\nu}=0)}(q)$ are the Fourier transforms of $\mathcal{L}_{\text{int}}(x)$ and $\mathcal{L}_{\text{int}}^{(B_{\mu\nu}=0)}(x)$ which is obtained by setting $B_{\mu\nu} = 0$ in (4.1), respectively. This relation can be generalized to the scattering amplitudes for any number of the incoming or outgoing physical particles. Thus the physical amplitude in this limit is given by the amplitude of the unphysical transverse photon, that is, the NG-boson of the spontaneous Lorentz symmetry breaking. This is an analogue of the equivalence theorem of the Yang-Mills-Higgs model. In the energy region $(B_{\text{vev}})^{\frac{2}{3}} \gg E \gg m$, where $(B_{\text{vev}})^{\frac{2}{3}}$ is the scale of the Lorentz symmetry breaking, the physical S-matrix elements can be obtained by the scattering amplitudes of the NG-bosons for the Lorentz symmetry breaking.

5 Discussions

In this paper, we have studied a covariant canonical quantization of a gauge invariant model of a second rank anti-symmetric tensor field and a vector field (photon). The spontaneous Lorentz symmetry breaking on the vacuum with a constant vev of the anti-symmetric tensor field has also been studied and the NG-boson of the Lorentz symmetry breaking has been identified with the unphysical photon. In this section, we discuss some related problems.

5.1 The spontaneous symmetry breaking of translation

Until now, we have discussed only the spontaneous Lorentz symmetry breaking on the vacuum (3.1). Indeed, the vacuum breaks the translational symmetry, because the vev

$\langle F_{12} \rangle = B_{\text{vev}}$ leads to, for example, the vev's

$$\langle A_1 \rangle = -\frac{1}{2}B_{\text{vev}}x^2 \quad \text{and} \quad \langle A_2 \rangle = \frac{1}{2}B_{\text{vev}}x^1. \quad (5.1)$$

Since A_1 and A_2 are not gauge invariant, one may expect that the vev's can be eliminated by a gauge transformation. However, following the discussion about the vev's of $B_{\mu\nu}$ and $F_{\mu\nu}$ in section 3, we do not require the invariance of the Hilbert space under the singular gauge transformation which eliminates them.

By the vev's (5.1), two translation generators P_1 and P_2 are broken:

$$\langle [iP_i, A_j(x)] \rangle = \epsilon_{ij} \frac{1}{2} B_{\text{vev}}, \quad (5.2)$$

where P_i is given by the canonical energy-momentum tensor (3.3)

$$P_i = \int d^2\mathbf{x} \, T_i^0(x). \quad (5.3)$$

The NG-bosons associated with the broken translation generator P_1 and P_2 are A_2 and A_1 respectively. The similar discussion to the broken Lorentz symmetry concludes that the NG-boson of the broken translational symmetry is also the unphysical massless photon.

This can be also understood from the following commutation relation in the Poincaré algebra¹¹:

$$[M_{0i}, P_0] = -iP_i. \quad (5.4)$$

Sandwiching the Jacobi identity

$$[[M_{0i}, P_0], A_j(x)] + [[A_j(x), M_{0i}], P_0] + [[P_0, A_j(x)], M_{0i}] = 0 \quad (5.5)$$

between the vacuum states $|\text{VAC}\rangle$, one can obtain the following equality

$$\langle [iP_i, A_j(x)] \rangle = \langle [iM_{0i}, \partial_0 A_j(x)] \rangle = \epsilon_{ij} \frac{1}{2} B_{\text{vev}}, \quad (5.6)$$

where we have used $P_0|\text{VAC}\rangle = 0$ and $[P_0, \phi_I] = -i\partial_0\phi_I$. This implies that when the NG-boson of the broken translation generator P_i is A_j , the NG-boson of the broken Lorentz generator M_{0i} is given by its time derivative $\partial_0 A_j \sim F_{0j}$. This phenomenon has been known as the inverse Higgs phenomenon in the nonlinear realization of space-time symmetries[19].

5.2 Relation to the nonlinear realization of Lorentz symmetry

According to the discussion in the previous subsection, the low-energy effective action of the NG-boson A_i can be obtained by the nonlinear realization of the translational and Lorentz symmetry, which leads to a 1-dimensional effective action¹⁵. However, since no physical lower-dimensional object such as a brane exist in our model, we expect a (2+1)-dimensional effective action which describes the low-energy effective theory in the whole space-time.

To realize this expectation, as in the case of the broken boost generators, we split the broken translation generators P_i into the broken parts and the unbroken parts such as $P_i = P_i^B(B_{\text{vev}}) + \hat{P}_i$ ¹⁶:

$$P_i^B = \frac{B_{\text{vev}}}{2} \int d^2 \mathbf{x} \, \epsilon_{ij} \left(\hat{U}^{0j} - \hat{C}^{0j} \right). \quad (5.7)$$

The broken parts of the translation generators satisfy

$$[iP_i^B, A_j] = \epsilon_{ij} \frac{1}{2} B_{\text{vev}}. \quad (5.8)$$

The same argument as the case of the boost generators leads us to the conclusion that the NG-bosons for the broken translation generators can be the unphysical massless photon.

Here, if one considers *only* the physical Hilbert space defined by the physical state conditions (2.16), one can show that the generators $\{M_{\mu\nu}, P_0, \hat{P}_i\}$ form a closed Poincaré algebra on the physical Hilbert space. As far as one considers this Poincaré algebra on the physical Hilbert space, the translational symmetry generated by \hat{P}_i is *unbroken* and only the boost symmetry generated by M_{0i} is broken. In this breaking pattern of the Poincaré symmetry, the low-energy effective action constructed by the nonlinear realization is a (2+1)-dimensional effective action and its explicit form has been obtained in [18]. Thus, we expect that the physical amplitudes of the model can be obtained by the low-energy effective action via the equivalence theorem discussed in the previous section.

5.3 Other related topics

From the relation between the non-commutative gauge theory and the D-brane world volume effective theory on a constant background anti-symmetric tensor field, our investigation is expected to give some new insights to the non-commutative gauge theory.

¹⁵The Nambu-Goto type effective action on a lower-dimensional brane embedded in higher dimensional flat space-time is known to be obtained by the nonlinear realization of the higher-dimensional translational and Lorentz symmetries [20].

¹⁶As in the case of the boost generators, one can show that P_i^B and \hat{P}_i are conserved separately.

However, in this context, the anti-symmetric tensor field is taken as an *external* field: the kinetic term for the anti-symmetric tensor field does not exist. Then the vacuum

$$\langle B_{12} \rangle \neq 0 \quad \text{and} \quad \langle F_{12} \rangle = 0, \quad (5.9)$$

is allowed as the solution of the equation of motion of a gauge field A_μ . This solution is a different vacuum from our vacuum (3.1). The extension of our analysis to the above case and the relation to the non-commutative gauge theory are interesting problems.

We make some speculations about the dynamics of photon A_μ in our model. At first, since the photon is the NG-boson of the broken Lorentz symmetry on the vacuum (3.1), the NG-theorem concludes that the photon cannot become massive on the vacuum. Secondly, as discussed in [18], the NG-boson of the broken Lorentz symmetry has derivative couplings with *any* fields including itself. Hence, the photon is expected to have derivative coupling with *any* fields including itself and neutral fields. From the relation to the non-commutative gauge theory, this seems consistent with the fact that the non-commutative photon in the non-commutative QED, where the products of fields are replaced by the \star -products, has self-couplings and derivative couplings with fields in the adjoint representation, i.e., neutral fields [21, 22].

Although the discussion in this paper is limited to the model in (2+1)-dimension, the similar discussions can be applied to the models in higher dimensions. Investigation of the more realistic case, that is, (3+1)-dimensional case is an interesting future problem.

Investigation of the supersymmetric extension of our model from the view point in this paper is also interesting.

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